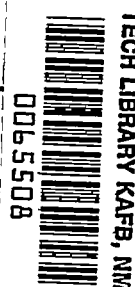


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TECHNICAL NOTE 2506

AN ANALYTIC DETERMINATION OF THE FLOW BEHIND
A SYMMETRICAL CURVED SHOCK

IN A UNIFORM STREAM

By C. C. Lin and S. F. Shen

Massachusetts Institute of Technology



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SUMMARY

The method of power-series expansion in solving the local flow pattern behind a detached shock was proposed by Lin and Rubinov. In this report the limitations of the method are discussed in a more careful manner, and the practical procedure for approximating the power series with a 2nth-degree polynomial by cutting off the remaining terms is investigated.

It is pointed out that if the power-series expansion is to hold near the nose, the body shape must be analytic up to and including the sonic point. For a 2nth-degree polynomial approximation, n parameters determining the shock shape, as well as the detached shock distance itself, are found to be expressible in terms of only $n - 1$ parameters determining the body shape. The formulas and steps for a sixth-degree polynomial approximation are explicitly given in an appendix.

The particular example of free-stream Mach number 1.7 in the axially symmetrical case has been worked out with a fourth-degree polynomial approximation. When the detached distance is used as the length scale, the flow along the axis is found to be independent of the body shape in this approximation. A comparison is made with data obtained from an interferometric study of the flow over a sphere. The density variation along the axis agrees very well. The detached distance as solved from the fourth-degree approximation, however, is correct only in the order of magnitude.

The universal density variation along the axis, as obtained by the fourth-degree approximation, is considered likely to be a good approximation for all bodies having a slowly varying curvature up to the sonic point at free-stream Mach numbers larger than or not much less than, say, the value 1.7 considered above.

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INTRODUCTION

It is well-known that a detached curved shock is formed in front of a body when a supersonic stream flows past it, provided the body has a blunt nose or a sharp nose with a quite large nose angle. A general treatment of such flow problems is difficult. It is therefore thought useful to try to obtain some results of limited applicability by straightforward methods of analysis. One such method is the use of power series proposed by one of the authors in reference 1 and partially carried out by Dugundji in reference 2. The present report is concerned with a closer investigation of the validity of this method and includes a more complete formulation, as well as the computation procedure, of the series expansion. General formulas were derived and some numerical calculations were carried out.

The case with free-stream Mach number 1.7 has been computed in the axisymmetrical case using a fourth-degree polynomial to represent the flow. As stated in reference 1, the flow thus determined depends only on the Mach number when the detached distance is adopted as the length scale. The density variation along the axis shows remarkable agreement with the experimental one for a sphere (reference 3), in spite of some indications of slow convergence of the series at the body nose. The effect of this poor convergence is manifest when an estimation of the detached distance itself is attempted. The discrepancy is large, although the order of magnitude is correct. This fact leads to the suspicion that perhaps the variation of density distribution does not depend very much on the exact shock shape and is therefore less sensitive to the inaccuracy of the method, yet the detached distance itself is very delicate. It is clear that more terms of the series are needed for a better agreement of the detached distance. General formulas and the procedure of calculation are given in this report for such purposes.

Because of the rather involved expressions for the series expansion and also the slow convergence occurring in the computed example, it is thought that the procedure here would be useful generally in the cases involving a body of approximately constant curvature in the neighborhood of the nose placed in a stream of Mach number not close to unity. One might notice that Busemann in reference 4 has stressed instead the importance of the "shoulder point," which presumably lies near the sonic point, in the determination of the detached shock in front of a body. His discussions, however, seem to deal mostly with thin bodies at rather low Mach numbers. A thin body with a blunt nose would have invariably a rather rapidly changing curvature near the nose. This configuration is then not directly amenable to the treatment by power series. (cf. discussions in the section "Discussion of Method.")

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SYMBOLS

ψ	stream function
α	angle between incoming stream and shock wave
γ	ratio of specific heats of gas
ρ	density of gas; also density ratio across shock wave
\bar{p}	pressure ratio across shock wave
δ	distance between shock wave and body along axis of symmetry
$F(\psi)$	dependence of entropy on streamline
M_∞	Mach number of incoming stream
Δs	increment of entropy across shock
β_1, β_2, \dots	shock shape parameters, defined by equations (8) and (13a) for axisymmetrical and two-dimensional cases, respectively
$\epsilon_1, \epsilon_2, \dots$	body shape parameters, defined by equation (21)
$\rho_0, \rho_1, \rho_2, \dots$	functions in expansion for ρ , defined by equations (15)
ψ_1, ψ_2, \dots	functions in expansion for ψ , defined by equations (15)
χ_0, χ_2, \dots	functions in another expansion for ψ , defined by equations (24)
ρ_{ij}, χ_{ij}	coefficients of Taylor series expansion of ρ and χ , respectively, defined by equation (26)
$\bar{\rho}_{ij}, \bar{\psi}_{ij}, \dots$	quantities evaluated at stagnation point

STATEMENT OF PROBLEM AND METHOD OF SOLUTION

Consider a symmetrical body with a blunt nose placed symmetrically in a uniform supersonic stream. It is plausible from physical grounds to assume that, for an analytic body shape, the detached shock would likewise be analytic.¹ The investigation will first be limited to such cases. This restriction evidently excludes some important problems, such as the detached shock formed in front of a cone or wedge, which must be treated separately.

To solve the problem, one may proceed in two steps. First, the shock curve may be regarded as given and the flow field regarded as determined by the initial conditions on the shock. If the shape of the shock is "reasonable," a stagnation point may be expected on the axis of symmetry, and a streamline may be found in the form of a body. The analysis may then be carried out by calculating the flow behind an analytic shock in the form of a power series. Strictly speaking, the analysis does not depend at all on the presence of the body and consists only of finding the relation between the shock shape and the flow behind it.

The second step is the investigation of the dependence of shock shape on body shape. Since the flow is determined by the shock curve and must also have the body as a streamline, the relation can be obtained by identifying the power-series solution from the shock with another power series developed around the nose of the body, the identification being made in a region common to the regions of convergence of both series. In fact, detailed calculations show that, by using series up to powers of the 2nth degree, this method enables one to solve for n "shock parameters," as well as the detached distance, in terms of $n - 1$ "body parameters." The parameters are in fact chosen to be the curvature and its successive derivatives evaluated at the initial point (nose). Although the method appears straightforward, careful investigations are needed to insure its validity. These will be carried out in the section "Discussion of Method."

The general analysis will be carried out in a manner applicable to both the two-dimensional and the axially symmetrical cases. Emphasis will then be put on the latter case in working out the explicit formulas and procedure.

¹In the actual case, there is a separation of the boundary layer often occurring in the supersonic region. The downstream shock will then cease to be analytic. But it is clear that this will produce no effect on the flow regime upstream of the separation point with which the present report is concerned.

The basic equations for flow behind a symmetrical shock produced in a uniform stream are, by introducing the conventional stream function ψ ,

$$\frac{1}{y^{2\epsilon}} \frac{1}{2\rho^2} (\psi_x^2 + \psi_y^2) + \frac{\gamma F(\psi) \rho^{\gamma-1}}{\gamma - 1} = C \quad (1)$$

$$\frac{\partial}{\partial x} \left(\frac{1}{\rho y^\epsilon} \frac{\partial \psi}{\partial x} \right) + \frac{\partial}{\partial y} \left(\frac{1}{\rho y^\epsilon} \frac{\partial \psi}{\partial y} \right) = - \frac{\rho^\gamma F'(\psi)}{\gamma - 1} y^\epsilon \quad (2)$$

where x is along the axis of symmetry, y is perpendicular to x , ρ is the density of the gas, γ , the ratio of specific heats, and C , Bernoulli's constant (the same in front of and behind the shock). The parameter ϵ is zero for the two-dimensional case but equals unity for the axially symmetrical case, and $F(\psi)$ gives the dependence of entropy on the streamline. To reduce equation (1) to dimensionless form, the free-stream values of velocity and density may be chosen to be the respective reference quantities. The length scale is arbitrary for the moment. Then it follows that

$$\left. \begin{aligned} F(\psi) &= (\gamma M_\infty^2)^{-1} e^{\Delta s} \\ C &= \frac{1}{2} + \frac{1}{(\gamma - 1) M_\infty^2} \end{aligned} \right\} \quad (3)$$

where M_∞ is the free-stream Mach number and Δs is the change of entropy across the shock (in multiples of the specific heat at constant volume). In fact, if α is the angle which the shock makes with the free stream,

$$e^{\Delta s} = \left(\frac{2\gamma}{\gamma + 1} M_\infty^2 \sin^2 \alpha - \frac{\gamma - 1}{\gamma + 1} \right) \left(\frac{\gamma - 1}{\gamma + 1} + \frac{2}{(\gamma + 1) M_\infty^2 \sin^2 \alpha} \right)^\gamma \quad (4)$$

The pressure ratio \bar{w} across the shock is

$$\bar{w} = \frac{2\gamma}{\gamma + 1} M_\infty^2 \sin^2 \alpha - \frac{\gamma - 1}{\gamma + 1} \quad (5)$$

and the density ratio across the shock is the dimensionless density

$$\rho = \left(\frac{\gamma - 1}{\gamma + 1} + \frac{2}{(\gamma + 1) M_\infty^2 \sin^2 \alpha} \right)^{-1} \quad (6)$$

It is easily seen that

$$\left. \begin{aligned} e^{\Delta s} &= \bar{w} \rho^{-\gamma} \\ F(\psi) &= (\gamma M_\infty^2)^{-1} \bar{w} \rho^{-\gamma} \end{aligned} \right\} \quad (7)$$

Although formulas (4) to (6) hold only along the shock, the dependence of $(\gamma M_1^2)^{-1} \bar{w} \rho^{-\gamma}$ on ψ , as given by equation (7), is valid throughout the whole field. One can therefore evaluate $F(\psi)$ from the shape of the shock curve.

Shock conditions.— Suppose now the shock curve is given by

$$\begin{aligned} z &= y^2 \\ &= \beta_1 x + \beta_2 x^2 + \beta_3 x^3 + \dots \end{aligned} \quad (8)$$

Then it can be easily verified that

$$\begin{aligned} (\sin \alpha)^{-2} &= 1 + 4z / [z'(x)]^2 \\ &= 1 + 4\beta_1^{-2} z - 16\beta_2\beta_1^{-4} z^2 - \\ &\quad 8\beta_1^{-2} (3\beta_3\beta_1^{-3} - 8\beta_2^2\beta_1^{-4}) z^3 + \dots \end{aligned}$$

$$\begin{aligned}
 (\sin \alpha)^2 &= [1 + 4z/(z^*)^2]^{-1} \\
 &= 1 - 4\beta_1^{-2}z + 16(\beta_2 + 1)\beta_1^{-4}z^2 + \\
 &\quad 8\beta_1^{-2}[3\beta_3\beta_1^{-3} + 8(\beta_2 + 1)^2\beta_1^{-4}]z^3 + \dots
 \end{aligned}$$

By straightforward calculations from equations (5), (6), and (7), one then obtains

$$\begin{aligned}
 \bar{\omega} &= \bar{\omega}_n \left\{ 1 + c_1\beta_1^{-2}z + c_2\beta_1^{-4}(\beta_2 + 1)z^2 + c_3[3\beta_1\beta_3 - \right. \\
 &\quad \left. 8(\beta_2 + 1)^2]\beta_1^{-6}z^3 + \dots \right\} \\
 \rho^{-1} &= \rho_n^{-1} \left[1 + c_1\beta_1^{-2}z + c_2\beta_1^{-4}\beta_2 z^2 + c_3(3\beta_1\beta_3 - 8\beta_2^2)z^3 + \dots \right] \\
 F &= F_n \left(1 + (\gamma c_1 + c_1)\beta_1^{-2}z + \left\{ (\gamma c_2 + c_2)\beta_2 + \right. \right. \\
 &\quad \left. \left[c_2 + \gamma c_1 c_1 + \frac{\gamma(\gamma - 1)}{2} c_1^2 \right] \beta_1^{-4}z^2 + \right. \\
 &\quad \left. \left\{ c_3[3\beta_1\beta_3 - 8(\beta_2 + 1)^2] + \gamma c_1 c_2(\beta_2 + 1) + \right. \right. \\
 &\quad \left. \gamma c_1 \left(\frac{\gamma - 1}{2} c_1^2 + c_2\beta_2 \right) + \gamma c_3(3\beta_1\beta_3 - 8\beta_2^2) + \right. \\
 &\quad \left. \left. \gamma(\gamma - 1)c_1 c_2 \beta_2 + \frac{\gamma(\gamma - 1)(\gamma - 2)}{6} c_1^3 \right\} \beta_1^{-6}z^3 + \dots \right)
 \end{aligned} \tag{9}$$

where the subscript n denotes quantities immediately behind the nose of the shock,

$$\left. \begin{aligned} \tilde{w}_n &= \frac{2\gamma}{\gamma+1} M_\infty^2 - \frac{\gamma-1}{\gamma+1} \rho_n^{-1} = \frac{\gamma-1}{\gamma+1} + \frac{2}{(\gamma+1)M_\infty^2} \\ F_n &= (\gamma M_\infty^2)^{-1} \left(\frac{2\gamma}{\gamma+1} M_\infty^2 - \frac{\gamma-1}{\gamma+1} \right) \left(\frac{\gamma-1}{\gamma+1} + \frac{2}{(\gamma+1)M_\infty^2} \right)^\gamma \end{aligned} \right\} \quad (10)$$

and the coefficients c_1 , c_2 , c_3 , C_1 , C_2 , and C_3 depend only on the Mach number of the undisturbed stream:

$$\left. \begin{aligned} C_1 &= -8\gamma / [2\gamma - (\gamma-1)M_\infty^{-2}] \\ C_2 &= -4C_1 \\ C_3 &= -2C_1 \\ c_1 &= 8 / [(\gamma-1)M_\infty^2 + 2] \\ c_2 &= -4c_1 \\ c_3 &= -2c_1 \end{aligned} \right\} \quad (11)$$

Equation (9) holds only on the shock. Hence, z may be replaced by a function of x through equation (8). For example, one obtains

$$\rho = \rho_n \left[1 - c_1 \beta_1^{-1} x + (c_1^2 + 3c_1 \beta_2) \beta_1^{-2} x^2 + \dots \right] \quad (12)$$

All the above calculations apply to the conditions at the shock both for the two-dimensional case and for the case of axial symmetry. The transformation of the third relation in equation (9) into one

involving ψ differs in the two cases. For the two-dimensional case, on the shock,

$$\begin{aligned}\psi &= y \\ &= \sqrt{z} \\ &= (\beta_1 x + \beta_2 x^2 + \beta_3 x^3 + \dots)^{1/2}\end{aligned}\quad (13a)$$

and hence

$$\begin{aligned}F(\psi) = F_n \bigg(1 + (\gamma c_1 + c_1) \beta_1^{-2} \psi^2 + \bigg\{ (\gamma c_2 + c_2) \beta_2 + \\ \left[c_2 + \gamma c_1 c_1 + \frac{\gamma(\gamma - 1)}{2} c_1^2 \right] \beta_1^{-4} \psi^4 + \left\{ c_3 [3\beta_1 \beta_3 - 8(\beta_2 + 1)^2] + \right. \\ \gamma c_1 c_2 (\beta_2 + 1) + \gamma c_1 \left(\frac{\gamma - 1}{2} c_1^2 + c_2 \beta_2 \right) + \gamma c_3 (3\beta_1 \beta_3 - 8\beta_2^2) + \\ \left. \gamma(\gamma - 1) c_1 c_2 \beta_2 + \frac{\gamma(\gamma - 1)(\gamma - 2)}{6} c_1^3 \right\} \beta_1^{-6} \psi^6 + \dots \bigg) \end{aligned}\quad (14a)$$

throughout the whole field of flow behind the shock. In the case of axial symmetry,

$$\begin{aligned}\psi &= \frac{1}{2} y^2 \\ &= \frac{1}{2} z \\ &= \frac{1}{2} (\beta_1 x + \beta_2 x^2 + \beta_3 x^3 + \dots)\end{aligned}\quad (13b)$$

on the shock, and hence

$$\begin{aligned}
 F(\psi) = F_n \bigg(1 + 2(\gamma c_1 + C_1) \beta_1^{-2} \psi + 4 \bigg\{ (\gamma c_2 + C_2) \beta_2 + \\
 \left[C_2 + \gamma c_1 C_1 + \frac{\gamma(\gamma - 1)}{2} c_1^2 \right] \beta_1^{-4} \psi^2 + 8 \bigg\{ C_3 [3\beta_1 \beta_3 - 8(\beta_2 + 1)^2] + \\
 \gamma c_1 C_2 (\beta_2 + 1) + \gamma c_1 \left(\frac{\gamma - 1}{2} c_1^2 + c_2 \beta_2 \right) + \gamma c_3 (3\beta_1 \beta_3 - 8\beta_2^2) + \\
 \gamma(\gamma - 1) c_1 c_2 \beta_2 + \frac{\gamma(\gamma - 1)(\gamma - 2)}{6} c_1^3 \bigg\} \beta_1^{-6} \psi^6 + \dots \bigg) \quad (14b)
 \end{aligned}$$

throughout the whole field of flow behind the shock. Equations (12), (13), and (14), with parameters defined by equations (10) and (11), are the basic relations associated with the shock curve (8).

Differential equations.—To solve differential equations (1) and (2) in power series, it is found convenient to develop ψ and ρ first in powers of y with functions of x as coefficients. From symmetry considerations, one has

$$\left. \begin{aligned} \psi &= y^{1+\epsilon} \left[\psi_1(x) + \psi_2(x)y^2 + \dots \right] \\ \rho &= \rho_0(x) + \rho_1(x)y^2 + \rho_2(x)y^4 + \dots \end{aligned} \right\} \quad (15)$$

Introducing these expressions into equations (1) and (2) and comparing powers of y , one obtains two series of differential equations for the two sets of functions $\psi_n(x)$ and $\rho_n(x)$. The function $F(\psi)$ is given by equation (14).

For the actual calculations, the two-dimensional case and the axially symmetrical case are best to be treated separately. Only the axisymmetrical case will be carried out in detail. Again introducing the variable $z = y^2$, one transforms equations (1) and (2) into

$$\frac{1}{2} \left(\frac{\psi_x^2}{z} + 4\psi_z^2 \right) + \frac{\gamma}{\gamma - 1} F(\psi) \rho^{\gamma+1} = C \rho^2 \quad (16)$$

$$\frac{\partial}{\partial x} \left(\frac{1}{\rho z} \psi_x \right) + 4 \frac{\partial}{\partial z} \left(\frac{1}{\rho} \psi_z \right) = - \frac{\rho^\gamma}{\gamma - 1} F'(\psi) \quad (17)$$

Equation (15) may be written as

$$\left. \begin{aligned} \psi &= \psi_1(x)z + \psi_2(x)z^2 + \psi_3(x)z^3 + \dots \\ \rho &= \rho_0(x)z + \rho_2(x)z^2 + \rho_3(x)z^3 + \dots \end{aligned} \right\} \quad (18)$$

Introducing equation (14b) in the abbreviated form

$$F(\psi) = a_0 + a_1\psi + a_2\psi^2 + a_3\psi^3 + \dots \quad (19)$$

and substituting equations (18) and (19) into equations (16) and (17) one derives the two sets of differential equations:

$$(A_0) \quad 4\psi_1^2 + \frac{2\gamma}{\gamma - 1} \rho_0^{\gamma+1} a_0 = 2C\rho_0^2$$

$$(A_1) \quad (\psi_1')^2 + 16\psi_1\psi_2 + \frac{2\gamma}{\gamma - 1} \rho_0^{\gamma+1} \left[a_0(\gamma + 1) \frac{\rho_1}{\rho_0} + a_1\psi_1 \right] = 4C\rho_0\rho_1$$

$$(A_2) \quad 2\psi_1'\psi_2' + 4(4\psi_2^2 + 6\psi_1\psi_3) + \frac{2\gamma}{\gamma - 1} \rho_0^{\gamma+1} \left\{ a_0 \left[(\gamma + 1) \frac{\rho_2}{\rho_0} + \frac{\gamma(\gamma + 1)}{2} \left(\frac{\rho_1}{\rho_0} \right)^2 \right] + a_1\psi_1(\gamma + 1) \frac{\rho_1}{\rho_0} + a_1\psi_2 + a_2\psi_1^2 \right\} = 2C(\rho_1^2 + 2\rho_0\rho_2)$$

$$\begin{aligned}
 (A_3) \quad & (\psi_2')^2 + 2\psi_1'\psi_3' + 4(12\psi_2\psi_3 + 8\psi_1\psi_4) + \frac{2\gamma}{\gamma-1} \rho_0^{\gamma+1} \left\{ a_0 \left[(\gamma+1) \frac{\rho_3}{\rho_0} + \right. \right. \\
 & \left. \left. \gamma(\gamma+1) \frac{\rho_1\rho_2}{\rho_0^2} + \frac{\gamma(\gamma^2-1)}{6} \frac{\rho_1^3}{\rho_0^3} \right] + a_1\psi_1 \left[(\gamma+1) \frac{\rho_2}{\rho_0} + \frac{\gamma(\gamma+1)}{2} \frac{\rho_1^2}{\rho_0^2} \right] + \right. \\
 & \left. (a_1\psi_2 + a_2\psi_1^2)(\gamma+1) \frac{\rho_1}{\rho_0} + a_1\psi_3 + 2a_2\psi_1\psi_2 + a_3\psi_1^3 \right\} = 4C(\rho_0\rho_3 + \rho_1\rho_2)
 \end{aligned}$$

$$(B_0) \quad \frac{d}{dx} \left(\frac{\psi_1'}{\rho_0} \right) + 4 \frac{\psi_1}{\rho_0} \left(\frac{2\psi_2}{\psi_1} - \frac{\rho_1}{\rho_0} \right) = -\frac{a_1}{\gamma-1} \rho_0^\gamma$$

$$\begin{aligned}
 (B_1) \quad & \frac{d}{dx} \left[\frac{\psi_1'}{\rho_0} \left(\frac{\psi_2'}{\psi_1'} - \frac{\rho_1}{\rho_0} \right) \right] + 8 \frac{\psi_1}{\rho_0} \left(\frac{3\psi_3}{\psi_1} - \frac{\rho_1}{\rho_0} \frac{2\psi_1}{\psi_2} + \frac{\rho_1^2}{\rho_0^2} - \frac{\rho_2}{\rho_0} \right) = \\
 & -\frac{\rho_0^\gamma}{\gamma-1} \left(2a_2\psi_1 + \gamma a_1 \frac{\rho_1}{\rho_0} \right)
 \end{aligned}$$

$$\begin{aligned}
 (B_2) \quad & \frac{d}{dx} \left[\frac{\psi_1'}{\rho_0} \left(\frac{\psi_3'}{\psi_1'} - \frac{\rho_1}{\rho_0} \frac{\psi_2'}{\psi_1'} + \frac{\rho_1^2}{\rho_0^2} - \frac{\rho_2}{\rho_0} \right) \right] + \\
 & 12 \frac{\psi_1}{\rho_0} \left[4 \frac{\psi_4}{\psi_1} - 3 \frac{\rho_1}{\rho_0} \frac{\psi_3}{\psi_1} + 2 \left(\frac{\rho_1^2}{\rho_0^2} - \frac{\rho_2}{\rho_0} \right) \frac{\psi_2}{\psi_1} + 2 \frac{\rho_1\rho_2}{\rho_0^2} - \frac{\rho_1^3}{\rho_0^3} - \frac{\rho_3}{\rho_0} \right] = \\
 & -\frac{\rho_0^\gamma}{\gamma-1} \left[2a_2\psi_2 + 3a_3\psi_1^2 + 2\gamma a_2\psi_1 \frac{\rho_1}{\rho_0} + a_1 \left(\gamma \frac{\rho_2}{\rho_0} + \frac{\gamma(\gamma-1)}{2} \frac{\rho_1^2}{\rho_0^2} \right) \right]
 \end{aligned}$$

Equations (A_0) and (B_0) are obtained by equating coefficients of z^0 in equations (16) and (17), respectively. The other equations arise from coefficients of corresponding powers.

One notes that from equation (A_0) , ψ_1 can be expressed in terms of ρ_0 . Then from equations (A_1) and (B_0) , ψ_2 and ρ_1 can be expressed in terms of ρ_0 , ρ_0' , and ρ_0'' ; from equations (A_2) and (B_1) , ψ_3 and ρ_2 can be expressed in terms of ρ_0 , ρ_0' , ρ_0'' , ρ_0''' , ρ_0^{iv} , and so on.

Determination of solution from the shock curve.- It will now be explicitly shown how the solution can be calculated as a power series when an analytic shock curve is given. For definiteness, the discussion will be limited to expansion up to the following degrees:

$\rho_0(x)$, $\psi_1(x)$ up to the fourth degree

$\rho_1(x)$, $\psi_2(x)$ up to the second degree

$\rho_2(x)$, $\psi_3(x)$ up to the constant terms only

Referring to equation (18), one sees that the physical variables ρ , u , and v are expanded to a fourth-degree polynomial in x and y .

In general, there is a decrease of two degrees when one proceeds from (ρ_n, ψ_{n+1}) to (ρ_{n+1}, ψ_{n+2}) . This is directly related to the above discussion about expressing them in terms of $\rho_0(x)$ and its derivatives. Again, in doing so, the expansion of ρ , u , and v as power series of x and y includes all the terms up to a certain degree.

Returning to the case of the fourth degree, one has to determine five coefficients for each of $\rho_0(x)$, $\psi_1(x)$, three coefficients for each of $\rho_1(x)$, $\psi_2(x)$, and the two quantities $\rho_2(0)$ and $\psi_3(0)$. However, since the quantities ψ_1 , ρ_1 , ψ_2 , ρ_2 , and ψ_3 are expressible in terms of $\rho_0(x)$ and its derivatives (up to the proper orders), there are only five coefficients to determine. This is done by imposing conditions (12) and (13b) for ρ and ψ on the shock. These power series are compared with another form of the same development obtained by introducing equation (8) into equation (18); namely,

$$\left. \begin{aligned}
 \rho &= \rho_0(0) + \left[\rho_0'(0) + \beta_1 \rho_1(0) \right] x + \left[\frac{1}{2} \rho_0''(0) + \rho_1'(0) \beta_1 + \rho_1(0) \beta_2 + \right. \\
 &\quad \left. \rho_2(0) \beta_1^2 \right] x^2 + \dots \\
 \psi - \frac{1}{2} z &= \left[\psi_1'(0) \beta_1 + \psi_2(0) \beta_1^2 \right] x^2 + \left[\psi_1'(0) \beta_2 + \frac{1}{2} \psi_1''(0) \beta_1 + \right. \\
 &\quad \left. \psi_2'(0) \beta_1^2 + 2\psi_2(0) \beta_1 \beta_2 + \psi_3(0) \beta_1^3 \right] x^3 + \dots
 \end{aligned} \right\} (20)$$

Exactly five conditions are obtained involving only the quantities desired. The only geometrical parameters of the shock entering the problem here are β_1 and β_2 which appear in the above equations and also enter through a_0 , a_1 , and a_2 appearing in the differential equations. Thus, the calculations of the solutions up to the fourth degree involve only two geometrical parameters from the shock. This has already been discussed by Lin and Rubinov (reference 1, p. 122), but here the procedure is explicitly outlined.

As a general rule, for a $2n$ th-degree expansion of the flow, the quantities involved will be up to $\psi_{n+1}(0)$ and $\rho_n(0)$. There are consequently n conditions from ψ and $n+1$ conditions from ρ by imposing equation (20). Thus the $2n+1$ coefficients in the expansion can be uniquely determined. It is also noted that there are n shock parameters $\beta_1, \beta_2, \dots, \beta_n$ entering the problem.

Relation between body shape and shock curve.— One starts next to expand the solution about the stagnation point of the body. Except for a shift of the origin, the previous equations (16) to (19), together with the consequences in the form of the equations (A) and (B), evidently hold just as well in the present expansion. The observation that ψ_{n+1} and ρ_n can be expressed in terms of $\rho_0, \rho_0', \rho_0'', \dots, \rho_0^{(2n)}$ remains also true. Suppose now the body shape is given by

$$z = y^2 = \epsilon_1 x + \epsilon_2 x^2 + \epsilon_3 x^3 + \dots \quad (21)$$

Then the conditions that furnish the relations for the determination of the coefficients become

$$\left. \begin{aligned} \rho_0(0) &= \left(\frac{2\gamma}{\gamma-1} \frac{c}{F_n(\psi)} \right)^{\frac{1}{\gamma-1}} \\ \psi_1(0) &= 0 \end{aligned} \right\} \quad (22)$$

and (cf. equations (20)),

$$\begin{aligned} 0 &= \left[\psi_1'(0)\epsilon_1 + \psi_2(0)\epsilon_1^2 \right] x^2 + \left[\psi_1'(0)\epsilon_2 + \frac{1}{2} \psi_1''(0)\epsilon_1 + \right. \\ &\quad \left. \psi_2'(0)\epsilon_1^2 + 2\psi_2(0)\epsilon_1\epsilon_2 + \psi_3(0)\epsilon_1^3 \right] x^3 + \dots \end{aligned} \quad (23)$$

One might reason that in comparison with equations (20), the conditions from density variations along the body are missing, and therefore there are $n+1$ fewer conditions. Even with the one for $\rho_0(0)$ in equations (22), it seems that for $2n+1$ coefficients in the expansion only $n+1$ conditions are available, leaving a total of n arbitrary parameters. Although the result turns out to be true, the argument is not so simple. An important point is that instead of n body parameters according to the above argument, only $n-1$ parameters defining the body here enter the problem. This may be verified as follows. Rewriting

$$\begin{aligned} \psi &= y^{1+\epsilon} \chi(x, y^2) \\ &= y^{1+\epsilon} (\chi_0 + \chi_2 y^2 + \chi_4 y^4 + \dots) \end{aligned} \quad (24)$$

with $\chi(0,0) = 0$ because the origin is chosen at the stagnation point, one may transform equation (16) into

$$\rho^{\gamma-1} = \frac{c - \frac{1}{\rho^2} \left[y^2 (\chi_x^2 + \chi_y^2) + 2(1+\epsilon)y\chi_y\chi + \chi^2 \right]}{\frac{\gamma-1}{\gamma} F(\psi)} \quad (25)$$

Each term in the bracket of the right-hand side is seen to have a double zero at the origin. After successive differentiation, the highest derivative of X on the right-hand side appears to be at least one order lower than the highest order of ρ on the left-hand side. In other words, if one writes formally

$$\rho = \sum_i \sum_j \rho_{ij} x^j y^i \quad (26)$$

and similar expressions for the other variables, there follows

$$\rho_{ij} = G(x_{i-2,j+1}, x_{i,j-1}, \dots) \quad (27)$$

where the omitted terms are lower-order derivatives of X and ρ . Since ρ is even in y , the index i is always even and terms like $x_{i-1,j}$ do not exist. Needless to say, the subscripts can never become negative. When a representation using a $2n$ th-degree polynomial is made, all the ρ_{ij} 's up to $i + j = 2n$ are required, involving the following derivatives of X :

$$\left. \begin{array}{l} x_{01} \quad x_{02} \dots x_{0,2n-1} \\ x_{20} \quad x_{21} \dots x_{2,2n-3} \\ \vdots \\ x_{2n-4,0} \quad x_{2n-4,1} \dots x_{2n-4,3} \\ x_{2n-2,0} \quad x_{2n-2,1} \end{array} \right\} \quad (28)$$

a total of $n^2 + n - 1$ quantities. Among them equation (17) furnishes some relations. By means of transformation (24), it becomes

$$x_{xx} + x_{yy} - \frac{1}{\rho} (\rho_x x_x + \rho_y x_y) + \frac{1+\epsilon}{y} \left[2x_y - (1+\epsilon) \frac{\rho_y}{\rho} x \right] - \frac{x_y}{y^\epsilon} - (1-\epsilon^2) \frac{x}{y^{1+\epsilon}} + \frac{\rho^{\gamma+1} F'(\psi)}{(\gamma-1)y^{1-\epsilon}} = 0 \quad (29)$$

Then x_{ij} is obtainable by applying the operator $\frac{\partial^{i+j-2}}{\partial x^i \partial y^{j-2}}$ to equation (29). All the terms except the first two can readily be seen to consist of lower-order derivatives than x_{ij} , the derivatives of ρ being representable by lower-order derivatives as expressed in equation (27). Hence

$$x_{ij} = H(x_{i-2, j+2}, \text{lower-order derivatives})$$

As a result, all the x_{ij} 's are expressible in terms of the derivatives with respect to x only. A total of $2n - 1$ parameters (e.g., x_{01} , x_{02} , . . . , $x_{0, 2n-1}$) is sufficient for the determination of all the needed x_{ij} 's in a $2n$ th-degree polynomial representation for ρ .

The general expression of condition (23) is now

$$x_0 + x_2 \sum_{n=1}^{\infty} \epsilon_n x^n + x_4 \left(\sum_{n=1}^{\infty} \epsilon_n x^n \right)^2 + \dots = 0 \quad (30)$$

The coefficient associated with x^m will yield an equation involving $x_{2m,0}$ as the highest-order term of X and ϵ_m as the highest-order term of body shape. Equation (28) indicates that for a $2n$ th-degree expansion the highest-order term of the form $x_{2m,0}$ needed is $x_{2n-2,0}$. Hence, equation (30) will furnish useful conditions by equating the coefficients to zero for terms up to x^{n-1} , that is, a total of $n - 1$ conditions. There are thus n arbitrary parameters (as suspected) left in the determination of the expansion. But since only terms up to x^{n-1} are used in equation (30), there are involved but $n - 1$ parameters defining the body shape.

It remains now to show how the relation between the shapes of the shock curve and of the body may be found. As stated above, the expansions from the shock and from the body are matched at a point lying within the regions of convergence of both. In particular, to avoid the question of singularity which might occur inside the body contour, the stagnation point itself is arbitrarily chosen as the "matching point." Along the x -axis, the $2n$ th-degree expansion furnishes $2n + 1$ conditions on equating the successive derivatives at the matching point. The variables involved are: n parameters defining the shock curve ($\beta_1, \beta_2, \dots, \beta_n$), n parameters in the expansion from the body (e.g., $\psi_{11}, \psi_{12}, \dots, \psi_{1n}$), $n - 1$ parameters defining the shape of the body ($\epsilon_1, \epsilon_2,$

. . . , ϵ_{n-1}) and the distance δ between the body and the detached shock wave along the x-axis. Hence a total of $2n + 1$ unknowns may be solved in terms of the $n - 1$ parameters defining the body shape. It may be noted that the solution obviously would be modified when different values of n are taken. However, if the series expansions do converge, the result should tend to a limit for increasing n .

If one does not care for a complete expansion in the form of equation (26) but rather is interested in the distribution along, say, the x-axis, the process is sometimes much simpler. By writing

$$\rho_0 = \rho_{10} + \rho_{01}\delta\left(\frac{x}{\delta}\right) + \rho_{02}\delta^2\left(\frac{x}{\delta}\right)^2 + \dots + \rho_{0,2n}\delta^{2n}\left(\frac{x}{\delta}\right)^{2n} \quad (31)$$

the coefficients $\rho_{0,m}\delta^m$ involve but a total of n parameters ($\beta_1, \beta_2, \dots, \beta_n$), with δ as the length scale. One could pick out from the set of $2n + 1$ simultaneous equations a subset which is sufficient for the solution of $\beta_1, \beta_2, \dots, \beta_n$. In general the subset would contain parameters occurring in the expansion from the body nose. In the particular example of a fourth-degree polynomial, the two necessary equations for β_1 and β_2 are readily provided by equation (22) or its equivalent and the expansion from the body is not needed. Evidently the body shape can not affect the fourth-degree expansion so determined. The simplicity of this result is certainly too attractive to be ignored, in spite of its possibly poor accuracy in extreme cases. The result turns out that a good agreement in density distribution is obtained with the experiment over a sphere at $M_\infty = 1.7$. Indication will be made in the section "Discussion of Method" as to the possible range over which a similar agreement might be expected to hold. When one tries to obtain the value of δ by imposing other conditions in the matching process, however, the result for the fourth-degree expansion shows a large discrepancy in comparison with the same experiment quoted above. This fact is not surprising since the power series is cut off after only four terms and the error would be undoubtedly large when higher derivatives are taken.

Explicit formulas and procedure.— In the actual carrying out of the solution, equations (A) and (B) are to be expanded further into ascending powers of x . Equations expressing the relation among the quantities ρ_{ij} and ψ_{ij} are obtained by equating the coefficients of each power of x . These formulas and also the procedure to solve for the desired coefficients in the case of a sixth-degree polynomial representation of the flow along the x-axis of an axially symmetrical body are given in the appendix.

DISCUSSION OF METHOD

The method described above, though seemingly straightforward, cannot be relied on without a more careful examination as to its limitations. Two points must be considered. In the first place, the subsonic region just behind a detached shock is expected to depend on the entire body shape. This seems to make it difficult to treat the problem by power series, which depend on local properties. The second is the question of convergence of the series.

The first point is really the following question: To what extent is the flow field determined by the local properties of the body at the nose? Theoretically, if the body shape is analytic, the local properties of the body do determine the entire body shape and thereby determine the entire flow field including the shock. In fact, at some distance downstream there is a sonic line (CD in fig. 1), after which the flow becomes supersonic. For two-dimensional flows, Guderley has suggested (reference 5) that, based on Tricomi's study of a certain differential equation which is elliptic in part of the region and hyperbolic in the rest, there should be also a unique solution for the subsonic region behind the shock. He reasoned that the body shape should be given up to the point E, from which the Mach wave reaches the sonic point on the shock front. Any reasonable modification of the body downstream from this point will not influence the flow in the region ABCED. The same conclusion, when physically interpreted, seems to be equally valid in three-dimensional flows. Thus, if a sufficient number of terms are retained in the power series for the body shape to represent it over the part BCE, slightly beyond the sonic point, all the important parameters of the problem are known. If the body has a fairly blunt nose, a few terms would be sufficient. The method of series expansion may then be expected to work.

It is important to notice that the above statement applies only to cases where the body contour is analytic over BCE. If the sonic point were brought in by the presence of a sharp turn in the boundary, a Prandtl-Meyer expansion exists, and the flow field in the region ABCD would be determined in an entirely different manner.

In order to apply the above discussions to a given body, one must be able to estimate the location of the sonic point. In certain cases, this might be done, for instance, by the method suggested by Busemann for the determination of the shoulder point (reference 4). One might even suggest that the body shape should be more closely approximated in that neighborhood at the expanse of its nose. The important point is that, since the entire contour BCE affects the shock, certain judgment is needed in picking out the most significant parameter. When there is more or less constant curvature starting from the nose, and the detached distance is known to be small in comparison with the radius of curvature

at the nose, undoubtedly the nose curvature plays a predominant role. Local properties of the shock, such as the curvature at its nose, would be largely determined by the nose curvature of the body. On the other hand, in cases like those discussed in reference 4, one has a thin body with a blunt nose and a quite large detached distance. The body curvature changes at a considerable rate in the neighborhood of the nose. This rate, then, might have an effect of the same order as or even overshadowing that of the curvature itself. The flow near the shock, being far away from the nose, would depend on the over-all body shape BCE. The simple procedure of using only a few terms of the expansion depending only on the body nose curvature is obviously insufficient for any accuracy at all.

After having clarified the dependence of the flow on the body shape near the nose, one must take up the question of the convergence of the series. Because of the analytic nature of the flow in the subsonic region, it might be assumed that the power-series expansion from the point A on the shock is convergent up to the body, including the stagnation point B. The region of convergence of the series starting from B, on the other hand, depends on the nearest singularity lying within the body contour. In choosing the point common to the regions of convergence for matching the two series, one must try to be close to the point B. In the section "Statement of Problem and Method of Solution" the point has in fact been taken to be point B itself. Indeed, if the series from A and B are closely approximated by polynomials of the same degree, the matching of the two series gives completely identical results, no matter where the conditions are applied. In fact, the present procedure of applying the method may also be regarded as one of polynomial approximation, satisfying a finite number of conditions at the boundaries. Then instead of talking about convergence, one can say that the inaccuracy is due to the failure to satisfy all the conditions as required by the governing equations of motion. There are techniques for improving the (over-all) accuracy of approximations of this nature. It is likely that by utilizing some of those techniques (e.g., least-square error) the practical value of the present method could be greatly enhanced.

After assuming the convergence of the series representation of the flow variables, one still has to consider the error caused by the retention of only a finite number of terms. It is important to point out that the solution for the parameters changes with the degree of the polynomial chosen, although theoretically the values would converge as n increases. The appropriate degree n for a satisfactory result, therefore, can not be stated a priori. One could merely argue that, for a body shape with slowly varying curvature up to and slightly beyond the sonic point, the shock-wave curvature presumably would also vary rather slowly up to its sonic point and, consequently, a polynomial of a fairly low degree would be needed in such cases. The effect of the

Mach number on the convergence of the series is more difficult to visualize. The question strictly can only be settled through actual calculation.

To clarify this point further, the validity of the fourth-degree polynomial as the universal function for density variation between the detached shock and the body nose will now be discussed. As pointed out above, one must look into the neglected terms in the series expansion. Consider, say, a sixth-degree representation. The coefficients thus solved will now contain, besides the Mach number, the product $K^{(0)}\delta$, where $K^{(0)}$ is the body curvature and δ , the detached distance. When $K^{(0)}\delta$ is small, the modification in the coefficients of the first five terms, as compared with the fourth-degree representation, would also be small. By dimensional reasoning $K^{(0)}\delta$ obviously depends only on the Mach number. So a range of Mach numbers prevails over which the fourth-degree representation is very close to the first five terms of the sixth-degree one. A thorough investigation should include the variation of the coefficient of the last two terms in the sixth-degree representation, which would lead to another restriction on the Mach number to justify their omission. A common range of Mach numbers would presumably become available for the fourth-degree representation to hold. These steps, though desirable, were not taken in this report. Without going into details, it seems physically likely that the smallness of $K^{(0)}\delta$ would ensure the fourth-degree representation to be a reasonable one. For, the smallness of $K^{(0)}\delta$ means that, for a given body curvature $K^{(0)}$, δ must be small. A smaller δ very likely tends to improve the practical convergence of the power series. It should be noted that, in characterizing the body by a single parameter $K^{(0)}$, one implies that the body shape must not deviate appreciably from a parabolic one up to the sonic point.

If the criterion of small $K^{(0)}\delta$ turns out to be correct, evidently a larger Mach number is favorable for the approximation. Semiempirically, the smallness of $K^{(0)}\delta$ may be tentatively measured against the value in the experiment quoted above, to be discussed in the following section, where the measured density distribution agrees well with the fourth-degree representation.

NUMERICAL RESULTS OF FOURTH-DEGREE POLYNOMIAL APPROXIMATION

The case of free-stream Mach number 1.7 has been computed for an axially symmetrical body by using a fourth-degree polynomial as the

approximation to the power-series solution. The results indicate that calculations involving higher-order terms are desirable for the determination of the distance of detachment. The steps outlined in the appendix have been followed. Taking $\gamma = 1.405$, one may list the results as follows:

$$\begin{aligned}
 \rho_{01} &= 7.296\beta_1^{-1} \\
 \psi_{11} &= -2.384\beta_1^{-1} \\
 \rho_{10} &= -12.83\beta_1^{-2} \\
 \psi_{20} &= 2.384\beta_1^{-2} \\
 \psi_{12} &= -16.51\beta_1^{-2} \\
 \rho_{02} &= -28.60\beta_1^{-2} \\
 \rho_{11} &= -29.18\beta_2\beta_1^{-3} - 142.4\beta_1^{-3} \\
 \psi_{21} &= 4.768\beta_2\beta_1^{-3} + 53.63\beta_1^{-3} \\
 \psi_{30} &= -7.152\beta_2\beta_1^{-4} - 37.12\beta_1^{-4} \\
 \rho_{20} &= 58.61\beta_2\beta_1^{-4} + 184.9\beta_1^{-4} \\
 \psi_{13} &= -10.80\beta_2\beta_1^{-3} - 73.46\beta_1^{-3} \\
 \rho_{03} &= 33.04\beta_2\beta_1^{-3} + 390.7\beta_1^{-3} \\
 \psi_{22} &= 140.4\beta_2\beta_1^{-4} + 596.2\beta_1^{-4} \\
 \rho_{12} &= -344.1\beta_2\beta_1^{-4} - 1772\beta_1^{-4} \\
 \psi_{14} &= -119.7\beta_2\beta_1^{-4} - 550.0\beta_1^{-4} \\
 \rho_{04} &= -350.3\beta_2\beta_1^{-4} - 6806\beta_1^{-4}
 \end{aligned}$$

.

There actually was another set of these functions because ρ_{01} is solved from a quadratic equation. However, the other expression gives a negative value of ρ_{01} , contrary to the conception of a compression along the axis and is therefore abandoned.

Conditions (22) may now be used to solve simultaneously for the parameters β_1/δ and β_2 . The equations are:

$$0.479 = \rho_{01}\delta + \rho_{02}\delta^2 + \rho_{03}\delta^3 + \rho_{04}\delta^4 \quad (32a)$$

$$0 = 0.500 + \psi_{11}\delta + \psi_{12}\delta^2 + \psi_{13}\delta^3 + \psi_{14}\delta^4 \quad (33a)$$

After substitution equations (32a) and (33a) become

$$0.479 = 7.296 \frac{\delta}{\beta_1} - 28.60 \left(\frac{\delta}{\beta_1} \right)^2 + 390.7 \left(\frac{\delta}{\beta_1} \right)^3 - 6806 \left(\frac{\delta}{\beta_1} \right)^4 +$$

$$\left(33.04 - 350.3 \frac{\delta}{\beta_1} \right) \left(\frac{\delta}{\beta_1} \right)^3 \beta_2 \quad (32b)$$

$$0.500 = 2.384 \frac{\delta}{\beta_1} + 16.51 \left(\frac{\delta}{\beta_1} \right)^2 + 73.46 \left(\frac{\delta}{\beta_1} \right)^3 + 550.0 \left(\frac{\delta}{\beta_1} \right)^4 +$$

$$\left(10.80 + 119.7 \frac{\delta}{\beta_1} \right) \left(\frac{\delta}{\beta_1} \right)^3 \beta_2 \quad (33b)$$

By eliminating β_2 , a fifth-degree algebraic equation in δ/β_1 is obtained. Again one has to choose the proper root which corresponds to the physical problem. In the present case, the choice is made easy by comparing with the experiment over a sphere as presented in reference 3. The proper root is found to be

$$\left. \begin{aligned} \frac{\delta}{\beta_1} &= 0.0644 \\ \beta_2 &= 50.4 \end{aligned} \right\} \quad (34)$$

Consequently, the density variation along the axis may be written down as

$$\rho_0 = 2.192 + 0.4699 \frac{x}{\delta} - 0.1186 \left(\frac{x}{\delta}\right)^2 + 0.5493 \left(\frac{x}{\delta}\right)^3 - 0.4214 \left(\frac{x}{\delta}\right)^4 \quad (35)$$

Up to this degree of representation, the body shape does not enter.

Before making comparison with the experimental data in reference 3, it may be noted that the solution for δ/β_1 and β_2 , such as equation (34), depends on the conditions chosen for their determination. To illustrate this point, one may recall that, instead of the condition $\psi_1 = 0$ at the stagnation point, an equivalent condition is $d\rho_0/dx = 0$ at the stagnation point along the axis. In the simultaneous equation, equation (33a) is to be replaced by

$$0 = \rho_{01}\delta + 2\rho_{02}\delta^2 + 3\rho_{03}\delta^3 + 4\rho_{04}\delta^4 \quad (36a)$$

so that in parallel with equation (33b) one now has

$$0 = 7.296 \frac{\delta}{\beta_1} - 57.21 \left(\frac{\delta}{\beta_1}\right)^2 + 1172 \left(\frac{\delta}{\beta_1}\right)^3 - 27360 \left(\frac{\delta}{\beta_1}\right)^4 + \left(99.11 - 1401 \frac{\delta}{\beta_1}\right) \left(\frac{\delta}{\beta_1}\right)^3 \beta_2 \quad (36b)$$

The solution of equations (32b) and (36b) gives

$$\left. \begin{aligned} \frac{\delta}{\beta_1} &= 0.0697 \\ \beta_2 &= 47.5 \end{aligned} \right\} \quad (37)$$

as well as

$$\rho_0 = 2.192 + 0.5085 \frac{x}{\delta} - 0.1390 \left(\frac{x}{\delta}\right)^2 + 0.6633 \left(\frac{x}{\delta}\right)^3 - 0.5538 \left(\frac{x}{\delta}\right)^4 \quad (38)$$

The difference between the two sets of results is not surprising, because only a finite number of terms have been used in the series. The terms omitted have unequal effects on the approximations for ρ_0 , $d\rho_0/dx$, and ψ_1 . Equations (35) and (38) are both plotted in figure 2 together with the experimental points from reference 3. The curve for equation (35) gives a slightly smaller density increase than equation (38) throughout the range. The agreement with experiment is good except for points very near to the body. There the experimental value is much higher than the theoretical one assuming isentropic compression. The discrepancy is too large to be attributed, say, to the neglect of the viscosity effect in the theory. On the other hand, one may suspect that the accuracy of the experiment could be poorer in the neighborhood of the stagnation point, where a very small region of high density occurs (see fig. 29, reference 3). A slight misalignment of the apparatus with the instantaneous actual flow direction, for instance, may cause a quite noticeable change in the value evaluated from the photograph.

The condition $d\rho_0/dx$ is, in fact, the one which results from the general scheme of matching the expansion from the shock with the expansion from the nose. Such a scheme will now be worked out for the present example for the purpose of finding the detached shock distance δ . Let $\bar{\rho}$ and $\bar{\psi}$ denote the variables in the expansion from the body nose. Again, by following the steps outlined in the appendix, one obtains readily:

$$\bar{\rho}_{00} = \left(\frac{\gamma - 1}{\gamma} \frac{c}{a_0} \right)^{\frac{1}{\gamma-1}}$$

$$\bar{\rho}_{01} = 0$$

$$\bar{\rho}_{10} = -\frac{1}{D_1} \bar{\psi}_{11}^2$$

$$\bar{\psi}_{20} = -\bar{\psi}_{11} \epsilon_1^{-1}$$

$$\bar{\psi}_{12} = 4\bar{\psi}_{11} \epsilon_1^{-1} - \frac{a_1}{2(\gamma - 1)} \bar{\rho}_{00}^{\gamma+1}$$

$$\bar{\rho}_{02} = -\frac{4}{D_1} \bar{\psi}_{11}^2$$

$$\bar{\rho}_{03} = -\frac{8}{D_1} \bar{\psi}_{11} \left[4\bar{\psi}_{11} \epsilon_1^{-1} - \frac{a_1}{2(\gamma - 1)} \bar{\rho}_{00}^{\gamma+1} \right]$$

$$\bar{\rho}_{04} = -\frac{8}{D_1} \bar{\psi}_{11} \bar{\psi}_{13} - \frac{4}{D_1} \left[4\bar{\psi}_{11} \epsilon_1^{-1} - \frac{a_1}{2(\gamma - 1)} \bar{\rho}_{00}^{\gamma+1} \right]^2 - \frac{D_2}{D_1} \bar{\rho}_{02}^2$$

The matching then involves a set of five simultaneous equations for the five unknowns β_1 , β_2 , δ , $\bar{\psi}_{11}$, and $\bar{\psi}_{13}$ in terms of the body shape parameter ϵ_1 :

$$\left. \begin{aligned} \rho_n + \rho_{01}\delta + \rho_{02}\delta^2 + \rho_{03}\delta^3 + \rho_{04}\delta^4 &= \bar{\rho}_{00} \\ \rho_{01}\delta + 2\rho_{02}\delta^2 + 3\rho_{03}\delta^3 + 4\rho_{04}\delta^4 &= \bar{\rho}_{01}\delta \\ 2\rho_{02}\delta^2 + 6\rho_{03}\delta^3 + 12\rho_{04}\delta^4 &= 2\bar{\rho}_{02}\delta^2 \\ 6\rho_{03}\delta^3 + 24\rho_{04}\delta^4 &= 6\bar{\rho}_{03}\delta^3 \\ \rho_{04}\delta^4 &= \bar{\rho}_{04}\delta^4 \end{aligned} \right\} \quad (39)$$

Obviously the first two are equations (32a) and (36a) solved before for the parameters δ/β_1 and β_2 . Note also that $\bar{\psi}_{13}$ appears only in $\bar{\rho}_{04}$; therefore the first four of equations (39) are sufficient to determine β_1 , β_2 , δ , and $\bar{\psi}_{11}$. One may easily verify that

$$\frac{\delta}{\epsilon_1} = \frac{\bar{\rho}_{03}\delta^3}{8\bar{\rho}_{02}\delta^2} - \frac{a_1\delta^2}{2(\gamma-1)\bar{D}_1} \frac{\bar{\rho}_{00}^{\gamma+1}}{\bar{\rho}_{02}\delta^2} \bar{\psi}_{11}\delta \quad (40)$$

with $\bar{\psi}_{11}\delta = -\left(-\frac{\bar{D}_1}{4} \bar{\rho}_{02}\delta^2\right)^{1/2}$. The negative root is taken for $\bar{\psi}_{11}\delta$

because $\bar{\psi}_{11} \sim \frac{\partial u}{\partial x}$ at the stagnation point, which is negative in the physical problem. With solution (37), the detached shock distance turns out to be

$$\frac{\delta}{\epsilon_1} = 0.138 \quad (41)$$

In comparison with the data in reference 3, the value of ϵ_1 being equal to the diameter of the sphere, the approximate formula (41) gives

$$\delta = 1.1 \text{ mm}$$

while the experimental value is 1.94 millimeters. The discrepancy is certainly large, yet the order of magnitude is correct. Since only a polynomial of the fourth degree is used, giving a density distribution independent of body shape, this is perhaps the reasonable extent of agreement that can be expected. More careful calculations must involve terms of higher orders.

One may also be interested to see how the shape of the shock curve is being approximated by the above solution. With equations (37), the shock shape from the present approximation is

$$\left(\frac{y}{\delta}\right)^2 = 14.36\left(\frac{x}{\delta}\right) + 47.5\left(\frac{x}{\delta}\right)^2 + \dots \quad (42)$$

The curve (42) has been compared with the experimental one given in reference 3. The agreement is reasonable for small values of y/δ , but the deviation becomes appreciable for values of y/δ of the order of unity, as may be expected from the slow convergence of the series (35) and (38).

Massachusetts Institute of Technology
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APPENDIX

EXPLICIT FORMULAS AND PROCEDURE FOR SIXTH-DEGREE
 POLYNOMIAL APPROXIMATION OF FLOW ALONG
 AXIS OF AXIALLY SYMMETRICAL BODY

It is desired now to find an approximate solution in the form

$$\rho_0 = \rho_{00} + \rho_{01}x + \rho_{02}x^2 + \rho_{03}x^3 + \rho_{04}x^4 + \rho_{05}x^5 + \rho_{06}x^6$$

for the flow along the axis of an axially symmetrical body. As stated in the section "Statement of Problem and Method of Solution," equations (A) and (B) are to be expanded into ascending powers of x . The resulting equation from equating the coefficients of a certain power of x of equation (A₀), say, is to be denoted by a second subscript. For instance, (A₀₀) stands for the equation obtained by equating the coefficient of x^0 in equation (A₀). In a similar manner, equation (B₂₂) stands for the equation obtained by equating the coefficient of x^2 in equation (B₂), and so forth. By so doing one gets the following:

$$(A_{00}) \quad \text{An identity between } \psi_{10} = 1/2 \text{ and } \rho_{00} = \rho_n$$

$$(A_{01}) \quad D_1 \rho_{01} = -8\psi_{10}\psi_{11}$$

$$(A_{02}) \quad D_1 \rho_{02} = -4(2\psi_{10}\psi_{12} + \psi_{11}^2) - D_2 \rho_{01}^2$$

$$(A_{03}) \quad D_1 \rho_{03} = -8(\psi_{10}\psi_{13} + \psi_{11}\psi_{12}) - 2D_2 \rho_{01} \rho_{02} -$$

$$\frac{2\gamma}{\gamma - 1} a_0 c_3^{\gamma+1} \rho_{00}^{\gamma-2} \rho_{01}^3$$

$$(A_{04}) \quad D_1 \rho_{04} = -4(2\psi_{10}\psi_{14} + 2\psi_{11}\psi_{13} + \psi_{12}^2) - (2\rho_{01}\rho_{03} + \rho_{02}^2)D_2 - \\ \left(\frac{2\gamma}{\gamma-1} a_0 c_3^{\gamma+1} \rho_{00}^{\gamma-2}\right)(3\rho_{01}^2 \rho_{02}) - \frac{2\gamma}{\gamma-1} a_0 c_4^{\gamma+1} \rho_{00}^{\gamma-3} \rho_{01}^4$$

$$(A_{05}) \quad D_1 \rho_{05} = -8(\psi_{10}\psi_{15} + \psi_{11}\psi_{14} + \psi_{12}\psi_{13}) - \\ (2\rho_{01}\rho_{04} + 2\rho_{02}\rho_{03})D_2 - \\ \frac{2\gamma}{\gamma-1} a_0 \left[3c_3^{\gamma+1} \rho_{00}^{\gamma-2} (\rho_{01}^2 \rho_{03} + \rho_{01}\rho_{02}^2) + \right. \\ \left. 4c_4^{\gamma+1} \rho_{00}^{\gamma-3} \rho_{01}^3 \rho_{02} + c_5^{\gamma+1} \rho_{00}^{\gamma-4} \rho_{01}^5 \right]$$

$$(A_{06}) \quad D_1 \rho_{06} = -4(2\psi_{10}\psi_{16} + 2\psi_{11}\psi_{15} + 2\psi_{12}\psi_{14} + \psi_{13}^2) - \\ (2\rho_{01}\rho_{05} + 2\rho_{04}\rho_{02} + \rho_{03}^2)D_2 - \\ \frac{2\gamma}{\gamma-1} a_0 \left[c_3^{\gamma+1} \rho_{00}^{\gamma-2} (3\rho_{01}^2 \rho_{04} + 6\rho_{01}\rho_{02}\rho_{03} + \rho_{02}^3) + \right. \\ \left. c_4^{\gamma+1} \rho_{00}^{\gamma-3} (4\rho_{01}^3 \rho_{03} + 6\rho_{01}^2 \rho_{02}^2) + \right. \\ \left. c_5^{\gamma+1} \rho_{00}^{\gamma-4} (5\rho_{01}^4 \rho_{02}) + c_6^{\gamma+1} \rho_{00}^{\gamma-5} \rho_{01}^6 \right]$$

$$(A_{10}) \quad \psi_{11}^2 + 16\psi_{10}\psi_{20} + D_1 \rho_{10} = -\frac{2\gamma}{\gamma-1} a_1 \rho_{00}^{\gamma+1} \psi_{10}$$

$$(A_{11}) \quad 16(\psi_{10}\psi_{21} + \psi_{20}\psi_{11}) + D_1\rho_{11} =$$

$$-4\psi_{12}\psi_{11} - 2D_2\rho_{10}\rho_{01} - \frac{2\gamma}{\gamma-1} a_1 \left(c_1^{\gamma+1} \rho_{00}^{\gamma} \rho_{01} \psi_{10} + \rho_{00}^{\gamma+1} \psi_{11} \right)$$

$$(A_{12}) \quad D_1\rho_{12} = -6\psi_{11}\psi_{13} - 4\psi_{12}^2 - 16(\psi_{10}\psi_{22} + \psi_{11}\psi_{21} + \psi_{12}\psi_{20}) -$$

$$2D_2(\rho_{10}\rho_{02} + \rho_{11}\rho_{01}) - \frac{2\gamma}{\gamma-1} a_0 \left(3c_3^{\gamma+1} \rho_{00}^{\gamma-2} \rho_{01}^2 \rho_{10} \right) -$$

$$\frac{2\gamma}{\gamma-1} a_1 \left[\psi_{12} \rho_{00}^{\gamma+1} + c_1^{\gamma+1} \rho_{00}^{\gamma} \rho_{01} \psi_{11} + \right.$$

$$\left. \left(c_1^{\gamma+1} \rho_{00}^{\gamma} \rho_{02} + c_2^{\gamma+1} \rho_{00}^{\gamma-1} \rho_{01}^2 \right) \psi_{10} \right]$$

$$(A_{13}) \quad D_1\rho_{13} = -8\psi_{11}\psi_{14} - 12\psi_{12}\psi_{13} - 16(\psi_{10}\psi_{23} + \psi_{11}\psi_{22} +$$

$$\psi_{12}\psi_{21} + \psi_{13}\psi_{20}) - 2D_2(\rho_{10}\rho_{03} + \rho_{02}\rho_{11} + \rho_{01}\rho_{12}) -$$

$$\frac{2\gamma}{\gamma-1} a_0 \left[c_3^{\gamma+1} \rho_{00}^{\gamma-2} \left(6\rho_{01}\rho_{02}\rho_{10} + 3\rho_{11}\rho_{01}^2 \right) + \right.$$

$$c_4^{\gamma+1} \rho_{00}^{\gamma-3} \left(4\rho_{01}^3 \right) \left. \right] - \frac{2\gamma}{\gamma-1} a_1 \left[\rho_{00}^{\gamma+1} \psi_{13} + \right.$$

$$c_1^{\gamma+1} \rho_{00}^{\gamma} \rho_{01} \psi_{12} + \left(c_1^{\gamma+1} \rho_{00}^{\gamma} \rho_{02} + c_2^{\gamma+1} \rho_{00}^{\gamma-1} \rho_{01}^2 \right) \psi_{11} +$$

$$\left. \left(c_1^{\gamma+1} \rho_{00}^{\gamma} \rho_{03} + 2c_2^{\gamma+1} \rho_{00}^{\gamma-1} \rho_{01}\rho_{02} + c_3^{\gamma+1} \rho_{00}^{\gamma-2} \rho_{01}^3 \right) \psi_{10} \right]$$

$$\begin{aligned}
(A_{14}) \quad D_1 \rho_{14} = & -10\psi_{11}\psi_{15} - 16\psi_{12}\psi_{14} - 9\psi_{13}^2 - 16(\psi_{10}\psi_{24} + \psi_{11}\psi_{23} + \\
& \psi_{12}\psi_{22} + \psi_{13}\psi_{21} + \psi_{14}\psi_{20}) - 2D_2(\rho_{01}\rho_{13} + \rho_{02}\rho_{12} + \\
& \rho_{03}\rho_{11} + \rho_{04}\rho_{10}) - \frac{2\gamma}{\gamma-1} a_0 \left[c_3^{\gamma+1} \rho_{00}^{\gamma-2} (6\rho_{10}\rho_{01}\rho_{03} + \right. \\
& 3\rho_{02}^2 \rho_{10} + 6\rho_{01}\rho_{02}\rho_{11} + 3\rho_{01}^2 \rho_{12}) + \\
& c_4^{\gamma+1} \rho_{00}^{\gamma-3} (12\rho_{01}^2 \rho_{02}\rho_{10} + 4\rho_{01}^3 \rho_{11}) + \\
& \left. c_5^{\gamma+1} \rho_{00}^{\gamma-4} (5\rho_{01}^4) \right] - \frac{2\gamma}{\gamma-1} a_1 \left\{ \rho_{00}^{\gamma+1} \psi_{14} + \right. \\
& c_1^{\gamma+1} \rho_{00}^{\gamma} \rho_{01} \psi_{13} + (c_1^{\gamma+1} \rho_{00}^{\gamma} \rho_{02} + c_2^{\gamma+2} \rho_{00}^{\gamma-1} \rho_{01}^2) \psi_{12} + \\
& (c_1^{\gamma+1} \rho_{00}^{\gamma} \rho_{03} + 2c_2^{\gamma+1} \rho_{00}^{\gamma-1} \rho_{01} \rho_{02} + \\
& c_3^{\gamma+1} \rho_{00}^{\gamma-2} \rho_{01}^3) \psi_{11} + [c_1^{\gamma+1} \rho_{00}^{\gamma} \rho_{04} + \\
& c_2^{\gamma+1} \rho_{00}^{\gamma-1} (2\rho_{01}\rho_{03} + \rho_{02}^2) + c_3^{\gamma+1} \rho_{00}^{\gamma-2} (3\rho_{01}^2 \rho_{02}) + \\
& \left. c_4^{\gamma+1} \rho_{00}^{\gamma-3} \rho_{01}^4] \psi_{10} \right\}
\end{aligned}$$

$$\begin{aligned}
(A_{20}) \quad 2\psi_{11}\psi_{21} + 24\psi_{10}\psi_{30} + D_1 \rho_{20} = \\
-16\psi_{20}^2 - D_2 \rho_{10}^2 - \frac{2\gamma}{\gamma-1} a_1 \rho_{00}^{\gamma} (c_1^{\gamma+1} \rho_{10} \psi_{10} + \rho_{00} \psi_{20}) - \\
\frac{2\gamma}{\gamma-1} a_2 \rho_{00}^{\gamma+1} \psi_{10}^2
\end{aligned}$$

$$\begin{aligned}
(A_{21}) \quad 24\psi_{10}\psi_{31} + D_1\rho_{21} = & -4(\psi_{11}\psi_{22} + \psi_{12}\psi_{21}) - 32\psi_{20}\psi_{21} - 24\psi_{11}\psi_{30} - \\
& 2D_2(\rho_{20}\rho_{01} + \rho_{10}\rho_{11}) - \\
& \frac{2\gamma}{\gamma-1} a_0 c_3^{\gamma+1} \rho_{00}^{\gamma-2} (3\rho_{01}\rho_{10}^2) - \\
& \frac{2\gamma}{\gamma-1} a_1 \left[c_1^{\gamma+1} \rho_{00}^{\gamma} (\rho_{10}\psi_{11} + \rho_{11}\psi_{10} + \right. \\
& \left. \rho_{01}\psi_{20}) + 2c_2^{\gamma+1} \rho_{00}^{\gamma-1} \rho_{10}\rho_{01} + \rho_{00}^{\gamma+1} \psi_{21} \right] - \\
& \frac{2\gamma}{\gamma-1} a_2 (2\rho_{00}^{\gamma+1} \psi_{10}\psi_{11} + c_1^{\gamma+1} \rho_{00}^{\gamma} \rho_{01}\psi_{10}^2)
\end{aligned}$$

$$\begin{aligned}
(A_{22}) \quad D_1\rho_{22} = & -6\psi_{11}\psi_{23} - 8\psi_{12}\psi_{22} - 6\psi_{13}\psi_{21} - 16(2\psi_{20}\psi_{22} + \psi_{21}^2) - \\
& 24(\psi_{10}\psi_{32} + \psi_{11}\psi_{31} + \psi_{12}\psi_{30}) - D_2(2\rho_{10}\rho_{12} + \rho_{11}^2 + \\
& 2\rho_{01}\rho_{21} + 2\rho_{02}\rho_{20}) - \frac{2\gamma}{\gamma-1} a_0 \left[c_3^{\gamma+1} 3(\rho_{01}^2\rho_{20} + \rho_{10}^2\rho_{02} + \right. \\
& \left. 2\rho_{01}\rho_{10}\rho_{11}) + c_4^{\gamma+1} 6\rho_{01}^2\rho_{10}^2 \right] - \frac{2\gamma}{\gamma-1} a_1 \left\{ \left[c_1^{\gamma+1} \rho_{00}^{\gamma} \rho_{12} + \right. \right. \\
& \left. 2c_2^{\gamma+1} \rho_{00}^{\gamma-1} (\rho_{01}\rho_{11} + \rho_{10}\rho_{02}) + 3c_3^{\gamma+1} \rho_{01}^2\rho_{10} \right] \psi_{10} + \\
& (c_1^{\gamma+1} \rho_{00}^{\gamma} \rho_{11} + c_2^{\gamma+1} \rho_{00}^{\gamma-1} 2\rho_{10}\rho_{01}) \psi_{11} + \\
& c_1^{\gamma+1} \rho_{00}^{\gamma} \rho_{10}\psi_{12} + (c_1^{\gamma+1} \rho_{00}^{\gamma} \rho_{02} + c_2^{\gamma+1} \rho_{00}^{\gamma-1} \rho_{01}^2) \psi_{20} + \\
& \left. c_1^{\gamma+1} \rho_{00}^{\gamma} \rho_{01}\psi_{21} + \rho_{00}^{\gamma+1} \psi_{22} \right\} - \frac{2\gamma}{\gamma-1} a_2 \left[(c_1^{\gamma+1} \rho_{00}^{\gamma} \rho_{02} + \right. \\
& c_2^{\gamma+1} \rho_{00}^{\gamma-1} \rho_{01}^2) \psi_{10}^2 + 2c_1^{\gamma+1} \rho_{00}^{\gamma} \rho_{01}\psi_{10}\psi_{11} + \\
& \left. \rho_{00}^{\gamma+1} (2\psi_{10}\psi_{12} + \psi_{11}^2) \right]
\end{aligned}$$

$$\begin{aligned}
 (A_{30}) \quad 2\psi_{11}\psi_{31} + 32\psi_{10}\psi_{40} + D_1\rho_{30} = & -\psi_{21}^2 - 48\psi_{20}\psi_{30} - 2D_2\rho_{10}\rho_{20} - \\
 & \frac{2\gamma}{\gamma-1} a_0 \left(c_3^{\gamma+1} \rho_{00}^{\gamma-2} \rho_{10}^3 \right) - \\
 & \frac{2\gamma}{\gamma-1} a_1 \left[\left(c_1^{\gamma+1} \rho_{00}^{\gamma} \rho_{20} + \right. \right. \\
 & \left. c_2^{\gamma+1} \rho_{00}^{\gamma-1} \rho_{10}^2 \right) \psi_{10} + \\
 & \left. c_1^{\gamma+1} \rho_{00}^{\gamma} \rho_{10} \psi_{20} + \rho_{00}^{\gamma+1} \psi_{30} \right] - \\
 & \frac{2\gamma}{\gamma-1} a_2 \left(c_1^{\gamma+1} \rho_{00}^{\gamma} \rho_{10} \psi_{10}^2 + \right. \\
 & \left. 2\rho_{00}^{\gamma+1} \psi_{10} \psi_{30} \right) - \frac{2\gamma}{\gamma-1} a_3 \rho_{00}^{\gamma+1} \psi_{10}^3
 \end{aligned}$$

$$(B_{00}) \quad 2\rho_{00}\psi_{12} = \psi_{11}\rho_{01} - 8\rho_{00}\psi_{20} + 4\rho_{10}\psi_{10} - \frac{a_1}{\gamma-1} \rho_{00}^{\gamma+2}$$

$$\begin{aligned}
 (B_{01}) \quad 6\rho_{00}\psi_{13} = & 2\psi_{11}\rho_{02} - 8(\rho_{00}\psi_{21} + \rho_{01}\psi_{20}) + \\
 & 4(\rho_{10}\psi_{11} + \rho_{11}\psi_{10}) - \frac{a_1}{\gamma-1} c_1^{\gamma+2} \rho_{00}^{\gamma+1} \rho_{01}
 \end{aligned}$$

$$\begin{aligned}
 (B_{02}) \quad 12\rho_{00}\psi_{14} = & 3\psi_{11}\rho_{03} + 2\rho_{02}\psi_{12} - 3\rho_{01}\psi_{13} - 8(\rho_{00}\psi_{22} + \rho_{01}\psi_{21} + \\
 & \rho_{02}\psi_{20}) + 4(\rho_{10}\psi_{12} + \rho_{11}\psi_{11} + \rho_{12}\psi_{10}) - \\
 & \frac{a_1}{\gamma-1} (c_1^{\gamma+2} \rho_{00}^{\gamma+1} \rho_{02} + c_2^{\gamma+2} \rho_{00}^{\gamma} \rho_{01}^2)
 \end{aligned}$$

$$\begin{aligned}
 (B_{03}) \quad 20\rho_{00}\psi_{15} = & 4\psi_{11}\rho_{04} + 4\rho_{03}\psi_{12} - 8\rho_{01}\psi_{14} - 8(\rho_{00}\psi_{23} + \rho_{01}\psi_{22} + \\
 & \rho_{02}\psi_{21} + \rho_{03}\psi_{20}) + 4(\rho_{10}\psi_{13} + \rho_{11}\psi_{12} + \rho_{12}\psi_{11} + \\
 & \rho_{13}\psi_{10}) - \frac{a_1}{\gamma-1}(c_1^{\gamma+2}\rho_{00}^{\gamma+1}\rho_{03} + 2c_2^{\gamma+2}\rho_{00}^{\gamma}\rho_{01}\rho_{02} + \\
 & c_3^{\gamma+2}\rho_{00}^{\gamma-1}\rho_{01}^3)
 \end{aligned}$$

$$\begin{aligned}
 (B_{04}) \quad 30\rho_{00}\psi_{16} = & 5\psi_{11}\rho_{05} + 6\rho_{04}\psi_{12} + 3\rho_{03}\psi_{13} - 4\rho_{02}\psi_{14} - 15\rho_{01}\psi_{15} - \\
 & 8(\rho_{00}\psi_{24} + \rho_{01}\psi_{23} + \rho_{02}\psi_{22} + \rho_{03}\psi_{21} + \rho_{04}\psi_{20}) + \\
 & 4(\rho_{10}\psi_{14} + \rho_{11}\psi_{13} + \rho_{12}\psi_{12} + \rho_{13}\psi_{11} + \rho_{14}\psi_{10}) - \\
 & \frac{a_1}{\gamma-1}\left[c_1^{\gamma+2}\rho_{00}^{\gamma+1}\rho_{04} + c_2^{\gamma+2}\rho_{00}^{\gamma}(\rho_{02}^2 + 2\rho_{01}\rho_{03}) + \right. \\
 & \left. 3c_3^{\gamma+2}\rho_{00}^{\gamma-1}\rho_{01}^2\rho_{02} + c_4^{\gamma+2}\rho_{00}^{\gamma-2}\rho_{01}^4\right]
 \end{aligned}$$

$$\begin{aligned}
 (B_{10}) \quad 2\rho_{00}\psi_{22} = & \rho_{01}\psi_{21} + 2\rho_{10}\psi_{12} + \rho_{11}\psi_{11} - 24\rho_{00}\psi_{30} + 16\rho_{10}\psi_{20} + \\
 & 8\rho_{20}\psi_{10} - g_{10}h_{10} - \frac{2a_2}{\gamma-1}\rho_{00}^{\gamma+2}\psi_{10} - \frac{\gamma a_1}{\gamma-1}\rho_{00}^{\gamma+1}\rho_{10}
 \end{aligned}$$

$$\begin{aligned}
 (B_{11}) \quad 6\rho_{00}\psi_{23} = & 2\rho_{02}\psi_{21} + 6\rho_{10}\psi_{13} + 4\rho_{11}\psi_{12} + (2\rho_{12} + 8\rho_{20})\psi_{11} - \\
 & 24(\rho_{00}\psi_{31} + \rho_{01}\psi_{30}) + 16(\rho_{10}\psi_{21} + \rho_{11}\psi_{22}) + 8\rho_{21}\psi_{10} - \\
 & (g_{10}h_{11} + g_{11}h_{10}) - \frac{2a_2}{\gamma - 1}(\rho_{00}^{\gamma+2}\psi_{11} + \\
 & c_1^{\gamma+2}\rho_{00}^{\gamma+1}\rho_{01}\psi_{10} - \frac{\gamma a_1}{\gamma - 1}(\rho_{00}^{\gamma+1}\rho_{11} + c_1^{\gamma+1}\rho_{00}^{\gamma}\rho_{01}\rho_{10})
 \end{aligned}$$

$$\begin{aligned}
 (B_{12}) \quad 12\rho_{00}\psi_{24} = & -3\rho_{01}\psi_{23} + (2\rho_{02} + 16\rho_{10})\psi_{22} + (3\rho_{03} + 16\rho_{11})\psi_{21} + \\
 & 16\rho_{12}\psi_{20} + 12\rho_{10}\psi_{14} + 9\rho_{11}\psi_{13} + (6\rho_{12} + 8\rho_{20})\psi_{12} + \\
 & (3\rho_{13} + 8\rho_{21})\psi_{11} + 8\rho_{22}\psi_{10} - 24(\rho_{00}\psi_{32} + \rho_{01}\psi_{31} + \\
 & \rho_{02}\psi_{30}) - (g_{10}h_{12} + g_{11}h_{11} + g_{12}h_{10}) - \\
 & \frac{2a_2}{\gamma - 1} \left[\rho_{00}^{\gamma+2}\psi_{12} + c_1^{\gamma+2}\rho_{00}^{\gamma+1}\rho_{01}\psi_{11} + \right. \\
 & \left. (c_2^{\gamma+2}\rho_{00}^{\gamma}\rho_{01}^2 + c_1^{\gamma+2}\rho_{00}^{\gamma+1}\rho_{02})\psi_{10} \right] - \\
 & \frac{\gamma a_1}{\gamma - 1} \left[\rho_{00}^{\gamma+1}\rho_{12} + c_1^{\gamma+1}\rho_{00}^{\gamma}\rho_{01}\rho_{11} + \right. \\
 & \left. (c_2^{\gamma+1}\rho_{00}^{\gamma-1}\rho_{01}^2 + c_1^{\gamma+1}\rho_{00}^{\gamma}\rho_{02})\rho_{10} \right]
 \end{aligned}$$

$$\begin{aligned}
(B_{20}) \quad 2\rho_{00}\psi_{32} = & 2\rho_{10}\psi_{22} + \psi_{21}(\rho_{11} - 2\rho_{01}g_{10}) - 2\psi_{12}(\rho_{00}g_{10}^2 - \rho_{20}) - \\
& \psi_{11}(2g_{10}\rho_{11} - 3g_{10}^2\rho_{01} - \rho_{21} + 2\rho_{01}g_{20}) + \psi_{31}\rho_{01} - \\
& 12\left[4\psi_{40}\rho_{00} - 3\psi_{30}\rho_{10} - 2\psi_{20}(\rho_{10}g_{10} - \rho_{20}) - \right. \\
& \left. \psi_{10}(2g_{10}\rho_{20} - g_{10}^3\rho_{10} - \rho_{30})\right] - \\
& \frac{a_1}{\gamma - 1}(c_1^{\gamma}\rho_{00}^{\gamma+1}\rho_{20} + c_2^{\gamma}\rho_{00}^{\gamma}\rho_{10}^2) - \\
& \frac{2a_2}{\gamma - 1}(\rho_{00}^{\gamma+2}\psi_{20} + c_1^{\gamma}\rho_{00}^{\gamma+1}\rho_{10}\psi_{10}) - \frac{3a_3}{\gamma - 1}\rho_{00}^{\gamma+2}\psi_{10}^2
\end{aligned}$$

In the above expressions, a number of symbols are used for abbreviation. They are defined as follows: The g_{mn} 's are coefficients of the quantity ρ_m/ρ_0 :

$$\frac{\rho_m}{\rho_0} = \sum_{n=0}^{\infty} g_{mn} x^n$$

Hence

$$g_{m0} = \frac{\rho_{m0}}{\rho_{00}}$$

$$g_{m1} = \frac{\rho_{m1}}{\rho_{00}} - \frac{\rho_{m0}\rho_{01}}{\rho_{00}^2}$$

$$g_{m2} = \frac{\rho_{m2}}{\rho_{00}} - \frac{\rho_{m1}\rho_{01}}{\rho_{00}^2} + \frac{\rho_{m0}}{\rho_{00}}\left(\frac{\rho_{01}^2}{\rho_{00}^2} - \frac{\rho_{02}}{\rho_{00}}\right)$$

.

The C_n^m 's are the binomial coefficients such that

$$(1 + x)^m = \sum_{n=0}^{\infty} C_n^m x^n$$

that is,

$$C_n^m = \frac{m(m-1) \cdots (m-n+1)}{n!}$$

Finally the D_m 's and h_{mn} 's are simply abbreviations:

$$D_1 = \frac{2a_0^\gamma}{\gamma - 1} C_1^{\gamma+1} \rho_{00}^\gamma - 4C\rho_{00}$$

$$D_2 = \frac{2a_0^\gamma}{\gamma - 1} C_2^{\gamma+1} \rho_{00}^{\gamma-1} - 2C$$

$$h_{10} = 2\rho_{01}\psi_{11} + 8\rho_{10}\psi_{10}$$

$$h_{11} = 4\rho_{01}\psi_{12} + (4\rho_{02} + 8\rho_{10})\psi_{11} + 8\rho_{11}\psi_{10}$$

$$h_{12} = 6\rho_{01}\psi_{13} + (8\rho_{02} + 8\rho_{10})\psi_{12} + (6\rho_{03} + 8\rho_{11})\psi_{11} + 8\rho_{12}\psi_{10}$$

For an expansion starting from the shock, the conditions at the shock are given by identifying equations (12) and (13b) with equation (20). The equation obtained from equating the coefficients of a certain power of x in the condition for ρ , say, is now to be denoted by a superscript to ρ . Thus $\rho^{(2)}$ stands for the equation resulting from the coefficients of x^2 in the condition for ρ ; similarly, $\psi^{(2)}$ stands

for the same in the condition for ψ . Then the following are found to hold:

$$\left(\rho^{(1)}\right) \quad \rho_{01} + \rho_{10}\beta_1 = -c_1\rho_{00}\beta_1^{-1}$$

$$\left(\rho^{(2)}\right) \quad \rho_{02} + \beta_1\rho_{11} + \beta_1^2\rho_{20} = -\beta_2\rho_{10} + \rho_{00}\left[-c_1\beta_2\beta_1^{-2} + \left(c_1^2 - c_2\beta_2\right)\beta_1^{-2}\right]$$

$$\left(\rho^{(3)}\right) \quad \beta_1^2\rho_{21} + \beta_1^3\rho_{30} = -\rho_{03} - \beta_1\rho_{12} - \beta_2\rho_{11} - \beta_3\rho_{10} - 2\beta_1\beta_2\rho_{20} + \rho_{00}\left[-c_1\beta_3\beta_1^{-2} + 2\beta_2\left(c_1^2 - c_2\beta_2\right)\beta_1^{-3} - \left(c_1^3 - 2c_1c_2\beta_2 - 3c_3\beta_1\beta_3 + 8c_3\beta_0^2\right)\beta_1^{-3}\right]$$

$$\left(\psi^{(2)}\right) \quad \psi_{11} + \psi_{20}\beta_1 = 0$$

$$\left(\psi^{(3)}\right) \quad \psi_{21}\beta_1 + \psi_{30}\beta_1^2 = -\psi_{12} - \psi_{20}\beta_2$$

$$\left(\psi^{(4)}\right) \quad \psi_{31}\beta_1^2 + \psi_{40}\beta_1^3 = -\psi_{13} - \psi_{20}\beta_3 - \psi_{21}\beta_2 - \psi_{22}\beta_1 - 2\psi_{30}\beta_1\beta_2$$

One may proceed in the order indicated below to arrive at all the coefficients ρ_{00} , ρ_{01} , . . . , ρ_{06} . The right-hand side of each expression is the unknown which may be determined from the left-hand side in terms of quantities already obtained.

$$(A_{00}) \rightarrow \text{identity between } \psi_{10} \text{ and } \rho_{00}$$

$$\left. \begin{array}{l} (A_{01}) \\ (A_{10}) \\ (\rho^{(1)}) \\ (\psi^{(2)}) \end{array} \right\} \rightarrow \psi_{11}, \rho_{01}, \psi_{20}, \rho_{10} \text{ simultaneously}$$

$$(B_{00}) \rightarrow \psi_{12}$$

$$(A_{02}) \rightarrow \rho_{02}$$

$$\left. \begin{array}{l} (A_{11}) \\ (A_{20}) \\ (\rho^{(2)}) \\ (\psi^{(3)}) \end{array} \right\} \rightarrow \psi_{21}, \rho_{11}, \psi_{30}, \rho_{20}$$

$$(B_{01}) \rightarrow \psi_{13}$$

$$(A_{03}) \rightarrow \rho_{03}$$

$$(B_{10}) \rightarrow \psi_{22}$$

$$(A_{12}) \rightarrow \rho_{12}$$

$$(B_{02}) \longrightarrow \psi_{14}$$

$$(A_{04}) \longrightarrow \rho_{04}$$

$$\left. \begin{array}{l} (A_{21}) \\ (A_{30}) \\ (\rho(3)) \\ (\psi(4)) \end{array} \right\} \longrightarrow \psi_{31}, \quad \rho_{21}, \quad \psi_{40}, \quad \rho_{30}$$

$$(B_{11}) \longrightarrow \psi_{23}$$

$$(A_{13}) \longrightarrow \rho_{13}$$

$$(B_{20}) \longrightarrow \psi_{32}$$

$$(A_{22}) \longrightarrow \rho_{22}$$

$$(B_{03}) \longrightarrow \psi_{15}$$

$$(A_{05}) \longrightarrow \rho_{05}$$

$$(B_{12}) \longrightarrow \psi_{24}$$

$$(A_{14}) \longrightarrow \rho_{14}$$

$$(B_{04}) \longrightarrow \psi_{16}$$

$$(A_{06}) \longrightarrow \rho_{06}$$

As a result, for a fourth-degree expansion, one only has to go as far as equation (A_{04}) , solving a total of 16 relations. For a sixth-degree one, there are altogether 30 relations. The increase of amount of work gives an indication as to what must be expected for a finer approximation.

Next, for an expansion starting from the stagnation point, the same set of equations (A_{00}) to (B_{20}) still holds. The quantities will be denoted by $\bar{\rho}_{00}$, $\bar{\psi}_{11}$, \bar{D}_1 , and so forth. The conditions resulting from equation (23) take the same form as $(\psi^{(2)})$, $(\psi^{(3)})$, . . . , except that the β 's are to be replaced by the ϵ 's. But the conditions resulting from the density variation along the body are missing. The previous order for the successive solution of the unknowns may still be followed, while one arbitrary parameter must now be introduced in each set of the simultaneous equations like the set (A_{01}) , (A_{10}) , and $(\psi^{(2)})$. Thus, for a sixth-degree expansion one may take $\bar{\psi}_{11}$, $\bar{\psi}_{13}$, and $\bar{\psi}_{15}$ as the parameters and break up the simultaneous equations into separate ones. Meanwhile the fact $\bar{\psi}_{10} = 0$ greatly simplifies the computation. The necessary steps, in fact, are reduced to:

$$(A_{00}) \rightarrow \bar{\rho}_{00}$$

$$(A_{01}) \rightarrow \bar{\rho}_{01}$$

$$(A_{10}) \rightarrow \bar{\rho}_{10}$$

$$(\psi^{(2)}) \rightarrow \bar{\psi}_{20}$$

$$(B_{00}) \rightarrow \bar{\psi}_{12}$$

$$(A_{02}) \rightarrow \bar{\rho}_{02}$$

$$(A_{03}) \rightarrow \bar{\rho}_{03}$$

$$(A_{04}) \rightarrow \bar{\rho}_{04}$$

$$(B_{01}) \rightarrow \bar{\psi}_{21}$$

$$(A_{11}) \longrightarrow \bar{p}_{11}$$

$$(\psi^{(3)}) \longrightarrow \bar{\psi}_{30}$$

$$(A_{20}) \longrightarrow \bar{p}_{20}$$

$$(B_{10}) \longrightarrow \bar{\psi}_{22}$$

$$(B_{02}) \longrightarrow \bar{\psi}_{14}$$

$$(A_{05}) \longrightarrow \bar{p}_{05}$$

$$(A_{06}) \longrightarrow \bar{p}_{06}$$

Hence a total of only 8 steps are sufficient for a fourth-degree expansion, and 16 steps, for a sixth-degree one. Further, there is no longer the need to solve some of the equations simultaneously.

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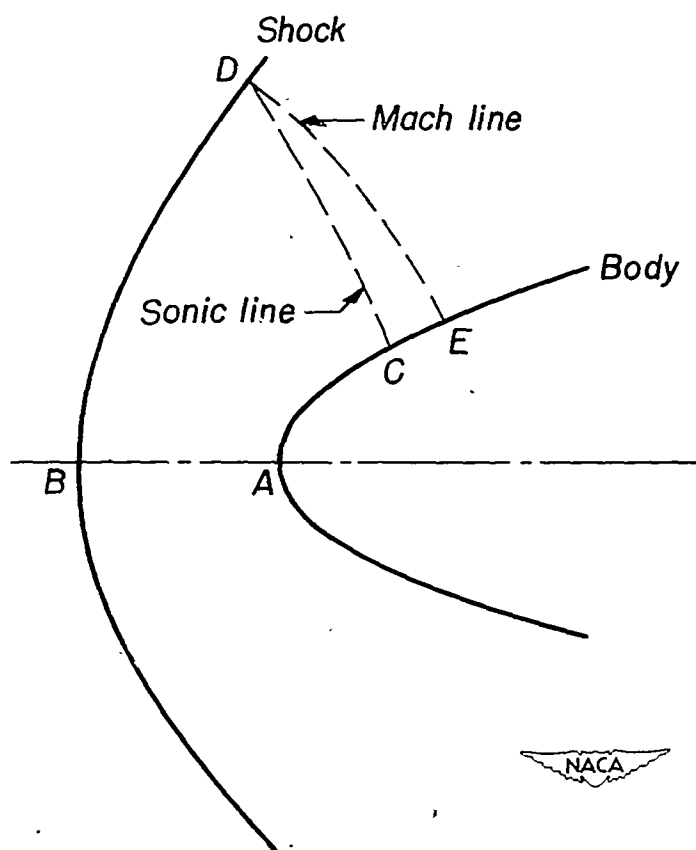


Figure 1.- Regions of flow behind a detached shock.

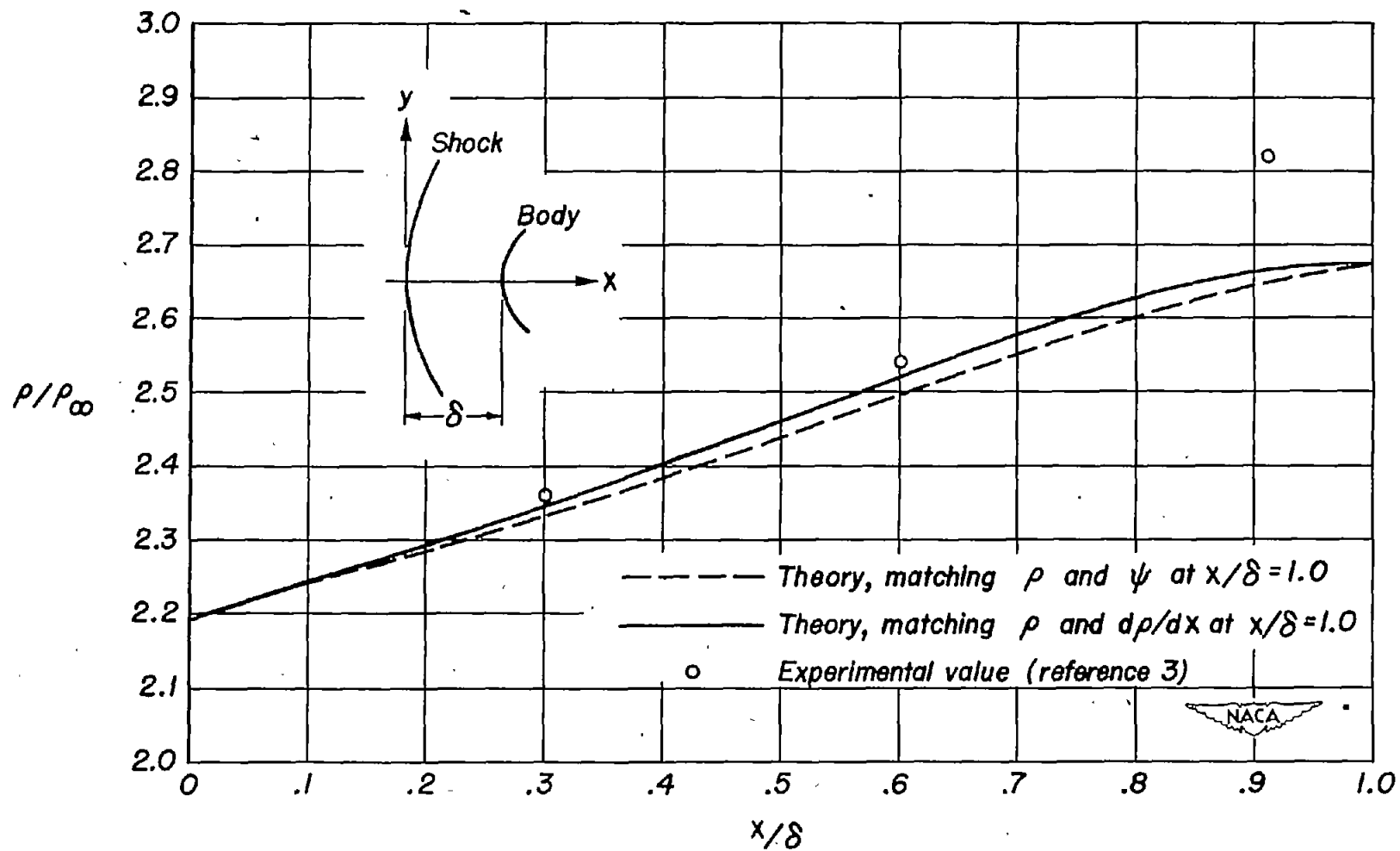


Figure 2.- Density variation along axis for $M_\infty = 1.7$.